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# Corner transfer matrices and corner magnetization for the Ising model

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**Abstract.** We consider the corner magnetization of an Ising square lattice with free edges along a diagonal direction. Using some remarkable properties of the corner transfer matrix, we give a unified treatment for various opening angles. The calculation of the magnetization is reduced to the solution of a matrix equation. In the thermodynamic limit we construct an explicit form of this equation and discuss its numerical and asymptotic solution and the results obtained thereby.

## 1. Introduction

The investigation of local order at corners of two-dimensional systems started only some years ago. It was first done for the Ising model [1] and gave an interesting result: the critical exponent of the corner magnetization ($m_c$) is inversely proportional to the geometrical angle (for isotropic systems) or an effective angle (for anisotropic systems) at the corner. Conformal arguments then showed that this is a general feature in two dimensions [1–3]. So far there exist results for several types of Ising model corners [1, 4, 5]. However, with the exception of the 180° corner [5, 6] (i.e. a straight surface), no completely analytical solution exists: the other results were obtained by using at least partly numerical methods. A magnetization formula for one particular 90° corner exists only as a conjecture [5]. This suggests that one should look for some new approach.

The method which we present in this paper is based on the corner transfer matrix (CTM) introduced by Baxter [7–9]. This quantity appears if one forms a two-dimensional system from a number of wedge-shaped segments. The CTM is the partition function of such a segment with the spins along the edges as variables. Fixing the spins along the outer boundary leads to a non-zero expectation value of the central spin. In this way bulk order parameters have been calculated especially for the more complicated solvable models [8, 10, 11]. In these calculations, the intriguingly simple eigenvalue spectrum of the CTM plays a crucial role. This spectrum has also been studied recently in connection with conformal invariance [12, 13].

Given its name, it may seem odd that the CTM so far has not been used for actual corner problems. However, there is a difficulty in this case, because the thermodynamic average implies an independent summation over all variables along the edges. Thus

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one first has to find a convenient way to handle this problem. The method we give is the following: first couple the two edges by a seam of additional bonds, thereby creating a conical system. Then the corner magnetization $m_c$ can be calculated from the usual simple trace formula [9] which, however, now contains the eigenvalues of the transfer matrix formed from the CTM *and* the seam operator. The original corner problem then is recovered as the (non-trivial) limit of zero bond strength in the seam. We show that in this limit only one eigenvalue contributes to $m_c$.

In principle, the calculation of this *single* eigenvalue involves the complete diagonalization of a more complicated object than the original CTM, followed by an equally difficult infinite limit. Due to a number of remarkable properties of the CTM—a series of 'mathematical flukes' [14]—it turns out that one has only to find a certain $N$-dimensional vector, $N$ being the linear size of the wedge. The matrix involved contains the fermion single-particle energies and eigenfunctions of the CTM. Whilst this paper treats the Ising model using fermion techniques, it should be remembered that the six- and eight-vertex models also have free-fermion CTMs. Moreover the power of the CTM in general stems from their quasiparticle spectra. So we appear to have another example of the privileged role played by corners and corner transfer matrices, which is known to lead to the occurrence of Virasoro characters in the spectra of CTMs of lattice models off criticality [15–18]. Certainly we have a very convenient and natural formulation for a corner magnetization problem. Especially, the effective opening angle $\theta_{\text{eff}}$ enters in a very simple way because of the property of CTMs that they are generated by exponentiation of a Hamiltonian boost operator [9] in the thermodynamic limit.

The paper is organized as follows. Section 2 reviews some general properties of the CTM and its diagonalization in terms of fermions. In section 3, the seam operator is introduced and the limit of zero bond strength is obtained. In section 4 the equations are further analysed and brought to a usable form. Section 5 contains the construction of the eigenvectors in the thermodynamic limit, leading to an explicitly stated set of equations which determine $m_c$. Section 6 gives numerical results and also discusses some asymptotics. Section 7 contains a summary and outlook.

## 2. General properties of the corner transfer matrices

The basic CTM—denoted herein either as $A(u, k)$ or more simply as $A$—is shown in figure 1 for a small lattice size. Since we will be concerned with magnetization problems we have added an extra line of spins at the edge, to fix the boundary conditions. The

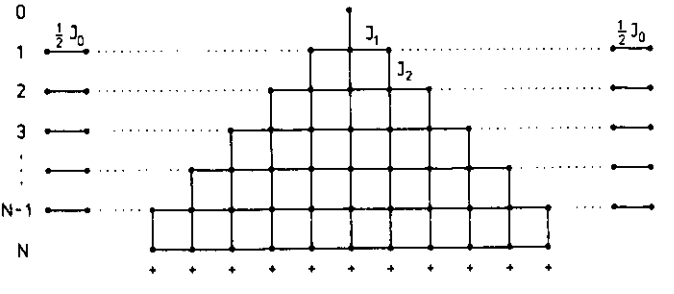


**Figure 1.** The corner geometry used in this paper. The wedge-shaped portion of the square lattice gives the corner transfer matrix $A$. To the left and right are half-seams of bonds connected by the dotted lines. The whole system gives the transfer matrix $T$ of equation (3.2).

effect of this is that there are two simple operators which commute with the CTM: they are $\sigma_0^x$ and $\sigma_N^x$ in the representation which we choose. The most important properties of the CTMs which we need depend only on the fermion algebra. Proofs of some of these were already given in [13], a paper to which shall refer as DP; here we only supplement those proofs as necessary.

The appropriate (elliptic) parametrization of the Boltzmann weights, using energies $J_i$ (normalized by the factor $kT$) is given in [19], viz

$$\sinh 2J_1 = -\mathrm{i}\,\mathrm{sn}(\mathrm{i}u) \qquad \sinh 2J_2 = \mathrm{i}\,\mathrm{ns}(\mathrm{i}u)/k \qquad k = (\sinh 2J_1 \sinh 2J_2)^{-1}. \tag{2.1}$$

The elliptic parameter $u$ is a measure of the anisotropy of the system, $k$ the elliptic modulus appropriate for the low temperature regime. We use the standard notation [20] for the modulus $k$, the complementary modulus $k'$ and the quarter periods $K$ and $K'$ of the elliptic functions. The interaction between two neighbouring spins defines an operator which may be represented in terms of Pauli matrices $\sigma_m^x$ and $\sigma_m^z$ which act on the spins in the $m$th line. We choose the representation in which $\sigma_m^x$ is diagonal: then the operator for an interaction of strength $J_1$ between two spins on the $m$th line is $X_{2m-1} = \rho_1 \exp(J_1^* \sigma_m^z)$, where $\rho_1$ is an unimportant normalization factor and $J_1^*$ is the dual of $J_1$. Similarly, the operator for an interaction of strength $J_2$ between two spins on the $m$th and $(m+1)$th lines is $X_{2m} = \exp(J_2 \sigma_m^x \sigma_{m+1}^x)$. Clearly the CTM is the product of such operators, one for each interaction: typical formulae are given in DP.

Fermion operators are introduced in the standard way via the Jordan–Wigner transformation:

$$c_m = P_{m-1}(\sigma_m^x - \mathrm{i}\sigma_m^y) \qquad c_m^\dagger = P_{m-1}(\sigma_m^x + \mathrm{i}\sigma_m^y) \qquad P_m = \prod_{n=0}^{m} (-\sigma_n^z). \tag{2.2}$$

Now it is shown in DP that, because $A$ is a product of operators $X_n$ which are spinor representations of rotations, it is possible in general to find a linear transformation of the $c_m$, $c_m^\dagger$ which transforms the CTM to a product of commuting operators of the same type:

$$A = \rho \prod_{j=0}^{N} \exp[-\gamma_j(\alpha_j^\dagger \alpha_j - \tfrac{1}{2})] \qquad \gamma_j > 0 \tag{2.3}$$

where $\rho$ is a normalization factor, and the transformation takes the form

$$\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^\dagger \end{pmatrix} = \mathscr{S} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix}. \tag{2.4}$$

The matrix $\mathscr{S}$, of dimension $2(N+1)$, together with the eigenvalues $\gamma_j$, are found by diagonalizing the matrix $\mathscr{A}$ defined by Heisenberg equations of motion (representation of the Clifford algebra) of the form

$$A \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix} A^{-1} = \mathscr{A} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix}. \tag{2.5}$$

In the present case, $\mathscr{A}$ is a real symmetric matrix, so there is a real orthogonal matrix $\mathscr{S}$ such that

$$\mathscr{S}\mathscr{A}\mathscr{S}' = \mathrm{diag}(\mathrm{e}^{\gamma_0}, \ldots, \mathrm{e}^{\gamma_N}, \mathrm{e}^{-\gamma_0}, \ldots, \mathrm{e}^{-\gamma_N}) \tag{2.6}$$

where diag stands for a diagonal matrix with the given entries. It is also shown in DP that the eigenvalues of $\mathscr{A}$ occur as real reciprocal pairs, whilst the eigenvectors are

paired under interchange of the creation and annihilation operators. Thus $S$ has a bipartite structure which we shall display by writing it in the block form

$$\mathcal{S} = \begin{pmatrix} \boldsymbol{G} & \boldsymbol{H} \\ \boldsymbol{H} & \boldsymbol{G} \end{pmatrix}. \tag{2.7}$$

$\mathcal{S}$ is formed by writing the components of the eigenvectors as its *rows*, because of the way it is defined in (2.6), and we write the rows in order of ascending magnitude of $\gamma_j$. There is some choice left for the zero eigenvalue which is doubly degenerate. We fix this by taking the first row of $\boldsymbol{G}$ as $(\frac{1}{2}, 0, \ldots, 0, \frac{1}{2})$ and the first row of $\boldsymbol{H}$ as $(\frac{1}{2}, 0, \ldots, 0, -\frac{1}{2})$. Since $\mathcal{S}$ is orthogonal, the matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ satisfy the relations

$$\boldsymbol{G}'\boldsymbol{G} + \boldsymbol{H}'\boldsymbol{H} = \boldsymbol{I} \qquad \boldsymbol{G}'\boldsymbol{H} + \boldsymbol{H}'\boldsymbol{G} = \boldsymbol{O}. \tag{2.8}$$

There is a further relation involving the columns of $\boldsymbol{G}$ and $\boldsymbol{H}$ which we will need. It comes from considering the identity $\mathcal{S}\mathcal{S}' = I$ in the positions where the zero eigenvectors are involved in the inner products. With the choices made above, we find that the first and last *columns* satisfy

$$\begin{aligned} \boldsymbol{G}_{j0} &= +\boldsymbol{H}_{j0} \quad j = 0 \qquad \boldsymbol{G}_{j0} = -\boldsymbol{H}_{j0} \quad j \neq 0 \\ \boldsymbol{G}_{jN} &= -\boldsymbol{H}_{jN} \quad j = 0 \qquad \boldsymbol{G}_{jN} = +\boldsymbol{H}_{jN} \quad j \neq 0. \end{aligned} \tag{2.9}$$

Now define a further matrix $\boldsymbol{E} = \operatorname{diag}(e^{\gamma_0}, \ldots, e^{\gamma_N})$, then equations (2.6)-(2.8) are equivalent to the block structure

$$\mathcal{A} = \begin{pmatrix} \boldsymbol{G}'\boldsymbol{E}\boldsymbol{G} + \boldsymbol{H}'\boldsymbol{E}^{-1}\boldsymbol{H} & \boldsymbol{G}'\boldsymbol{E}\boldsymbol{H} + \boldsymbol{H}'\boldsymbol{E}^{-1}\boldsymbol{G} \\ \boldsymbol{H}'\boldsymbol{E}\boldsymbol{G} + \boldsymbol{G}'\boldsymbol{E}^{-1}\boldsymbol{H} & \boldsymbol{H}'\boldsymbol{E}\boldsymbol{H} + \boldsymbol{G}'\boldsymbol{E}^{-1}\boldsymbol{G} \end{pmatrix}. \tag{2.10}$$

A most useful factorization may now be made. First define two further matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ by

$$\boldsymbol{C} = (\boldsymbol{E}^{1/2}\boldsymbol{G} \pm \boldsymbol{E}^{-1/2}\boldsymbol{H}) \qquad \boldsymbol{D} = (\boldsymbol{E}^{-1/2}\boldsymbol{G} \pm \boldsymbol{E}^{1/2}\boldsymbol{H}). \tag{2.11}$$

Then equation (2.10) takes the form

$$\mathcal{A} = \begin{pmatrix} \boldsymbol{C}'\boldsymbol{C} & \pm\boldsymbol{C}'\boldsymbol{D} \mp \boldsymbol{I} \\ \pm\boldsymbol{D}'\boldsymbol{C} \mp \boldsymbol{I} & \boldsymbol{D}'\boldsymbol{D} \end{pmatrix}. \tag{2.12}$$

This factorization will be central to our calculation of the magnetization in section 4.

## 3. Boundary conditions for a corner

The aim is to construct a transfer matrix whose trace contains the desired information about the corner magnetization with open boundary conditions. In the uses of CTMs in the existing literature, the system is in the form of a cone—the edges which form the corner are glued together. Consider now the seam operator for a line of bonds of strength $J_0$

$$W = \exp\left[J_0^* \sum_{m=1}^{N-1} \sigma_m^z\right] = \exp\left[\nu \sum_{m=1}^{N-1} (2c_m^\dagger c_m - 1)\right] \qquad \nu = J_0^*. \tag{3.1}$$

In the representation which we are using, this mixes together states with corresponding spin values, and those with opposing spins, with relative Boltzmann weights $\cosh \nu$ and $\sinh \nu$, respectively. An open boundary is the case that $\nu \to \infty$, a cone is the limit

$\nu \to 0$. Thus we want to diagonalize the product of two non-commuting operators, the CTM and the seam operator, and then take the limit $\nu \to \infty$. As usual for transfer matrices of product form, it is more convenient to consider the symmetrized product

$$T = W^{1/2} A W^{1/2}. \tag{3.2}$$

This is the form shown in figure 1, with bonds $\frac{1}{2}J_0$ on each side.

Now the seam operator (3.1), like the CTM itself, is a product of spinor representations of rotations, so the general principles for the diagonalization, and the properties of the matrices involved, are the same for $T$ as for $A$. The effect of the seam operator on the fermion operators is

$$W \begin{pmatrix} c \\ c^\dagger \end{pmatrix} W^{-1} = \mathcal{W} \begin{pmatrix} c \\ c^\dagger \end{pmatrix} \tag{3.3}$$

where

$$\mathcal{W} = \mathrm{diag}(1, \mathrm{e}^{-2\nu}, \ldots, \mathrm{e}^{-2\nu}, 1, 1, \mathrm{e}^{2\nu}, \ldots, \mathrm{e}^{2\nu}, 1). \tag{3.4}$$

Hence, the seam changes the problem of diagonalizing the matrix $\mathcal{A}$ to that of diagonalizing $\mathcal{T} = \mathcal{W}^{1/2} \mathcal{A} \mathcal{W}^{1/2}$.

Our goal in this section is to take the limit $\nu \to \infty$. In this limit, we expect that the seam operator will dominate the behaviour of the excitation spectrum. In fact the excitation spectrum of $W$ is simply that there are *two* zero-energy modes whilst all the other modes have energy $2\nu$. Therefore we expect that for $T$ the excitation spectrum has one zero-energy mode, one mode whose energy remains finite as $\nu \to \infty$, whilst all other modes have excitation energies which are of order $2\nu$. In fact, we have observed this behaviour in numerical calculations for small size systems. To extract the limit analytically, we (temporarily) write the matrix $\mathcal{A}$ with the rows and columns in a different order. We write first the four rows and columns which correspond to the entries 1 in $\mathcal{W}$, then we write those rows and columns which correspond to the entries $\exp(-2\nu)$, and finally those corresponding to the entries $\exp(2\nu)$. With this convention, the matrix $\mathcal{T}$ which we need to diagonalize is related to $\mathcal{A}$ by the following block structure

$$\mathcal{T} = \begin{pmatrix} \mathcal{A}_{00} & \mathrm{e}^{-\nu}\mathcal{A}_{0-} & \mathrm{e}^{\nu}\mathcal{A}_{0+} \\ \mathrm{e}^{-\nu}\mathcal{A}_{-0} & \mathrm{e}^{-2\nu}\mathcal{A}_{--} & \mathcal{A}_{-+} \\ \mathrm{e}^{\nu}\mathcal{A}_{+0} & \mathcal{A}_{+-} & \mathrm{e}^{2\nu}\mathcal{A}_{++} \end{pmatrix}. \tag{3.5}$$

Write the eigenvectors using the same block structure, then the eigenvalue equation is

$$\begin{pmatrix} \mathcal{A}_{00} & \mathrm{e}^{-\nu}\mathcal{A}_{0-} & \mathrm{e}^{\nu}\mathcal{A}_{0+} \\ \mathrm{e}^{-\nu}\mathcal{A}_{-0} & \mathrm{e}^{-\nu}\mathcal{A}_{--} & \mathcal{A}_{-+} \\ \mathrm{e}^{\nu}\mathcal{A}_{+0} & \mathcal{A}_{+-} & \mathrm{e}^{2\nu}\mathcal{A}_{++} \end{pmatrix} \begin{pmatrix} u_0 \\ u_- \\ u_+ \end{pmatrix} = \mathrm{e}^{\omega} \begin{pmatrix} u_0 \\ u_- \\ u_+ \end{pmatrix}. \tag{3.6}$$

We want solutions for which $\gamma$ remains finite as $\nu \to \infty$. The top block of (3.6) reads

$$\mathcal{A}_{00}u_0 + \mathrm{e}^{-\nu}\mathcal{A}_{0-}u_- + \mathrm{e}^{\nu}\mathcal{A}_{0+}u_+ = \mathrm{e}^{\omega}u_0 \tag{3.7}$$

whilst the other two blocks show that $u_-$ and $u_+$ are both of order $\exp(-\nu)$ compared with $u_0$. Consequently, only $u_+$ contributes to (3.7): the block which determines it reads

$$\mathrm{e}^{\nu}\mathcal{A}_{+0}u_0 + \mathcal{A}_{+-}u_- + \mathrm{e}^{2\nu}\mathcal{A}_{++}u_+ = \mathrm{e}^{\omega}u_+. \tag{3.8}$$

To leading order in $\exp(-\nu)$ the solution of (3.8) is

$$u_+ = -\mathrm{e}^{-\nu}\mathcal{A}_{0+}\mathcal{A}_{++}^{-1}\mathcal{A}_{+0}u_0 \tag{3.9}$$

and when this is substituted into (3.7) we obtain an eigenvalue equation for a $4\times4$ matrix $\mathfrak{A}$: viz

$$\mathfrak{A}\boldsymbol{u}_0=(\mathscr{A}_{00}-\mathscr{A}_{0+}\mathscr{A}_{++}^{-1}\mathscr{A}_{+0})\boldsymbol{u}_0=\mathrm{e}^{\omega}\boldsymbol{u}_0. \tag{3.10}$$

This is the main result of this section: together with the factorization (2.6) it gives, at least in principle, a method to calculate the only excitation energy which remains finite when the boundaries are open.

The formula for the corner magnetization may now be given in terms of the matrix $\mathfrak{A}$. In general, the magnetization formulae which are given by CTMs are infinite products of the form

$$\prod_{j=1}^{\infty}\tanh\tfrac{1}{2}\omega_j \tag{3.11}$$

involving the single fermion excitation energies $\omega_j$. In the present case, however, only one excitation makes any contribution to such a formula. Furthermore, we know from general principles that the eigenvalues of $\mathfrak{A}$ are 1, 1, $\mathrm{e}^{\omega}$, $\mathrm{e}^{-\omega}$, so we have trace $\mathfrak{A}=2(1+\cosh\omega)$. Then the formula $m_c=\tanh\frac{1}{2}\omega$ for the magnetization can be written as

$$m_c^2=1-\frac{4}{\operatorname{trace}\mathfrak{A}}. \tag{3.12}$$

## 4. Construction of the matrix $\mathfrak{A}$

The central problem for the construction of $\mathfrak{A}$ is the inversion of $\mathscr{A}_{++}$, followed by its use in equation (3.10). In this section we shall find a number of properties of $\mathfrak{A}$ which will reduce its construction to the computation of a single entry in the solution of a set of linear equations. The starting point is the factorization (2.12). There is a choice of sign in this factorization but the result will not depend on that choice. However, some of the intermediate details do have such a dependence, so in this section we set $\boldsymbol{C}=\boldsymbol{E}^{1/2}\boldsymbol{G}-\boldsymbol{E}^{-1/2}\boldsymbol{H}$ and $\boldsymbol{D}=\boldsymbol{E}^{-1/2}\boldsymbol{G}-\boldsymbol{E}^{1/2}\boldsymbol{H}$. We also define an $(N+1)\times(N-1)$ matrix $\boldsymbol{B}$ as the following restriction of $\boldsymbol{D}$,

$$\boldsymbol{B}_{jm}=\boldsymbol{D}_{jm}\qquad 0\leqslant j\leqslant N,\quad 0<m<N \tag{4.1}$$

for then we get the factorization

$$\mathscr{A}_{++}=\boldsymbol{B}'\boldsymbol{B}. \tag{4.2}$$

We also need those columns of $\mathscr{A}$ which form $\mathscr{A}_{+0}$. In the notation of equation (2.12) they are $(\boldsymbol{I}-\boldsymbol{D}'\boldsymbol{C})_{j0}$, $(\boldsymbol{I}-\boldsymbol{D}'\boldsymbol{C})_{jN}$, $(\boldsymbol{D}'\boldsymbol{D})_{j0}$ and $(\boldsymbol{D}'\boldsymbol{D})_{jN}$ with the restriction $0<j<N$. Hence the term involving $\boldsymbol{I}$ does not contribute and may be dropped and we see that all the columns are given by the action of $\boldsymbol{B}'$ on various vectors. Moreover, when we use the property (2.9) we find that the vectors involved satisfy $\boldsymbol{C}_{j0}=\boldsymbol{D}_{j0}$, $j>0$ and $\boldsymbol{C}_{jN}=-\boldsymbol{D}_{jN}$, $j>0$. Consequently, the construction of $\mathfrak{A}$ depends solely on entries from the lower right quarter $\boldsymbol{D}'\boldsymbol{D}$ of $\mathscr{A}$.

To simplify the use of this information, define three $(N+1)$-dimensional vectors $\boldsymbol{e}$, $\boldsymbol{u}$, $\boldsymbol{v}$ as

$$\begin{aligned}&\boldsymbol{e}=(1,0,\ldots,0)\\&\boldsymbol{u}_j=0\quad j=0\qquad \boldsymbol{u}_j=\boldsymbol{D}_{j0}\quad j>0\\&\boldsymbol{v}_j=0\quad j=0\qquad \boldsymbol{v}_j=\boldsymbol{D}_{jN}\quad j>0\end{aligned} \tag{4.3}$$

The entries of $\mathscr{A}_{00}$ may be written down immediately in terms of these vectors: viz

$$\mathscr{A}_{00} = \begin{pmatrix} \boldsymbol{u}.\boldsymbol{u} & -\boldsymbol{u}.\boldsymbol{v} & 1-\boldsymbol{u}.\boldsymbol{u} & -\boldsymbol{u}.\boldsymbol{v} \\ -\boldsymbol{v}.\boldsymbol{u} & 1+\boldsymbol{v}.\boldsymbol{v} & \boldsymbol{v}.\boldsymbol{u} & \boldsymbol{v}.\boldsymbol{v} \\ 1-\boldsymbol{u}.\boldsymbol{u} & \boldsymbol{u}.\boldsymbol{v} & \boldsymbol{u}.\boldsymbol{u} & \boldsymbol{u}.\boldsymbol{v} \\ -\boldsymbol{v}.\boldsymbol{u} & \boldsymbol{v}.\boldsymbol{v} & \boldsymbol{v}.\boldsymbol{u} & 1+\boldsymbol{v}.\boldsymbol{v} \end{pmatrix}. \tag{4.4}$$

Similarly, the entries of $\mathscr{A}_{0+}\mathscr{A}_{++}^{-1}\mathscr{A}_{+0}$ may be written in terms of inner products of the form $\boldsymbol{u}\boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{u}$. Consider such a product, and set

$$(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{u} = \boldsymbol{x}. \tag{4.5}$$

This is equivalent to the equations $\boldsymbol{B}'\boldsymbol{B}\boldsymbol{x} = \boldsymbol{B}'\boldsymbol{u}$ whose solution takes the form $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{u} + \alpha\boldsymbol{n}$ where $\boldsymbol{n}$ is a null vector of $\boldsymbol{B}'$. Given such a null vector, $\alpha$ is determined by the fact that the solution $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{u} + \alpha\boldsymbol{n}$ is itself an overdetermined set of equations, so it is necessary that the right-hand side belong to the column space of $\boldsymbol{B}$. Now $\boldsymbol{B}'$ has two less rows than columns, and hence two independent null vectors. Moreover, one of these is $\boldsymbol{e}$ and this certainly does not belong to the column space of $\boldsymbol{B}$, which has a zero top row. Denote the non-trivial normalized null vector by $\boldsymbol{n}$, then we find

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n} \tag{4.6}$$

from which it follows that

$$\boldsymbol{u}\boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})^2. \tag{4.7}$$

The result of this argument is that the matrix elements we need will have two contributions: inner products of the form $\boldsymbol{u}\cdot\boldsymbol{u}$ which will cancel the corresponding terms in (4.4), and the two inner products

$$\alpha = \boldsymbol{u}\cdot\boldsymbol{n} \qquad \beta = \boldsymbol{v}\cdot\boldsymbol{n}. \tag{4.8}$$

Collecting all this information, we find for $\mathfrak{A}$ the structure

$$\mathfrak{A} = \begin{pmatrix} \alpha^2 & -\alpha\beta & 1-\alpha^2 & -\alpha\beta \\ -\alpha\beta & 1+\beta^2 & \alpha\beta & \beta^2 \\ 1-\alpha^2 & \alpha\beta & \alpha^2 & \alpha\beta \\ -\alpha\beta & \beta^2 & \alpha\beta & 1+\beta^2 \end{pmatrix}. \tag{4.9}$$

We must determine the inner products $\alpha$ and $\beta$. For this purpose, consider the matrix

$$X = \begin{pmatrix} \boldsymbol{E}^{-1/2}\boldsymbol{G} & \boldsymbol{E}^{1/2}\boldsymbol{H} \\ \boldsymbol{E}^{1/2}\boldsymbol{H} & \boldsymbol{E}^{-1/2}\boldsymbol{G} \end{pmatrix}. \tag{4.10}$$

Its inverse $X^{-1}$ has the block structure

$$X^{-1} = \begin{pmatrix} \tilde{\boldsymbol{G}} & \tilde{\boldsymbol{H}} \\ \tilde{\boldsymbol{H}} & \tilde{\boldsymbol{G}} \end{pmatrix} \tag{4.11}$$

whilst the transpose satisfies

$$X'X = \begin{pmatrix} \boldsymbol{D}'\boldsymbol{D} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D}'\boldsymbol{D} \end{pmatrix}. \tag{4.12}$$

It follows that $(X'X)^{-1} = X^{-1}(X^{-1})'$ is block diagonal: expressed in terms of $(\tilde{\boldsymbol{G}}, \tilde{\boldsymbol{H}})$ we obtain

$$\tilde{\boldsymbol{G}}\tilde{\boldsymbol{H}}' + \tilde{\boldsymbol{H}}\tilde{\boldsymbol{G}}' = \boldsymbol{O} \qquad \tilde{\boldsymbol{G}}\tilde{\boldsymbol{G}}' + \tilde{\boldsymbol{H}}\tilde{\boldsymbol{H}}' = (\boldsymbol{D}'\boldsymbol{D})^{-1}. \tag{4.13}$$

The first relation is orthogonality between rows from different blocks.

Equation (4.11) may be written in terms of $\tilde{G}$ and $\tilde{H}$ as two independent equations:

$$(\tilde{G} \pm \tilde{H})(E^{-1/2}G \pm E^{1/2}H) = I. \tag{4.14}$$

Consequently, the rows 0 and $N$ of $(\tilde{G} - \tilde{H})$ contain the null vector we need, in a combination to be determined. In fact, formulation of the problem using $X$ gives null vectors for both possible definitions of $D$ in (2.11). In either case, one null vector is $e$ and we want to find the other. Now define two vectors $w$ and $w'$ by writing the relevant rows (in an obvious notation) as

$$\begin{aligned} (\tilde{G})_0 &= \tfrac{1}{2}(e + w) \qquad & (\tilde{H})_0 &= \tfrac{1}{2}(e - w) \\ (\tilde{G})_N &= \tfrac{1}{2}(e - w') \qquad & (\tilde{H})_N &= -\tfrac{1}{2}(e + w'). \end{aligned} \tag{4.15}$$

Orthogonality of rows from different blocks of $X$ gives the important results

$$|w| = 1 \qquad |w'| = 1 \qquad w \cdot e = w' \cdot e. \tag{4.16}$$

We shall use these to obtain the formula (4.20) below expressing the magnetization $m_c$ as the component $w_0$ of $w$. However, the fact that $w_0 = w_0'$ means that the formula is unchanged if the other sign is chosen in the factorizaton of $\mathcal{A}$, resulting in an interchange of the roles of $w$ and $w'$. Therefore we do not refer to $w'$ again.

Now we can evaluate the inner products $\alpha$, $\beta$, as follows. From (4.14) we know that four inner products between various vectors have a value of either 0 or 1. Only two of these are needed, namely

$$w \cdot u = 1 \qquad w \cdot (e + v) = 0. \tag{4.17}$$

The null vector $n$ must be orthogonal to $e$, so it has the form $n = \lambda(w - w_0 e)$, where $\lambda$ is a normalization factor. Using this in the definitions (4.8), together with (4.17), we find

$$\alpha = \lambda \qquad \beta = -\lambda w_0. \tag{4.18}$$

From (4.16) we know that $w$ is normalized, so we find that

$$\lambda^2 = 1/(1 - w_0^2) \qquad \alpha^2 - \beta^2 = 1. \tag{4.19}$$

The foregoing argument is quite long, but the conclusion is quite simple. The matrix $\mathfrak{A}$ has the structure (4.9), and the necessary inner products, which are related, are given by (4.18). The result becomes even more simple when we use it in (3.12) to obtain the magnetization. After a straightforward calculation we find that

$$m_c = |w_0|. \tag{4.20}$$

The equations which determine $w$ are more easily analysed by following Lieb *et al* [21] and introducing new matrices $\Phi$ and $\Psi$ as

$$\Phi = G + H \qquad \Psi = G - H. \tag{4.21}$$

If we use the notation that the entries of $\Phi$ and $\Psi$ in the $j$th row—the eigenvectors for $\pm\gamma_j$—are written as $\phi_j(m)$ and $\psi_j(m)$, then the equation for $w$ is

$$\sum_{j=0}^{N} w_j[\cosh(\tfrac{1}{2}\gamma_j)\psi_j(m) - \sinh(\tfrac{1}{2}\gamma_j)\phi_j(m)] = \delta_{m,0}. \tag{4.22}$$

*This is a central equation of this paper: its solution gives the corner magnetization* directly through (4.20). There is an important variation for the solution of (4.22) which follows from the special structure of the zero eigenvectors. The top row of the coefficient

matrix has the entries $(0, \ldots, 0, 1)$, hence we may first solve the equations indexed by $0 \leqslant m < N$, for the components of $w_j$ indexed by $1 \leqslant j \leqslant N$, independently of the value of $w_0$. Then $w_0$ may be found either by back substitution, or we may simply use the fact that $w$ is normalized to find $|w_0|$. The second method is more convenient for taking the thermodynamic limit; the former for calculations with systems of finite size.

## 5. Thermodynamic limit

Implementation of the method requires that we construct the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ for use in (4.22). For numerical calculation on small systems this can be done by constructing and diagonalizing the matrix $\mathscr{A}$ directly from its definition in terms of spinor rotations. Since our chief interest is in the thermodynamic limit, we prefer an explicit construction: for this we use the property that the CTM is generated exactly by exponentiation of a Hamiltonian boost operator in the limit of large $N$ [9, 13]. This is why we introduced the elliptic parametrization in (2.1): in these variables we have $A(u, k) = a \exp[-uH(k)]$, where $a$ is an unimportant normalization factor and

$$H = -k \sum_{m=1}^{N-1} m(2c_m^\dagger c_m - 1) - \sum_{m=0}^{N-1} (m+\tfrac{1}{2})(c_m^\dagger - c_m)(c_{m+1}^\dagger + c_{m+1}). \tag{5.1}$$

Diagonalization is a standard problem in fermion algebra [19, 22]: we therefore give only sufficient detail for the present purpose. The analogue of (2.5) for the Hamiltonian is

$$\left[\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix}, H\right] = \mathscr{H}\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix} \tag{5.2}$$

and then the matrix $\mathscr{S}$ of eigenvectors, and the eigenvalues $\lambda_j > 0$, are given by

$$\mathscr{S}\mathscr{H}\mathscr{S}' = \operatorname{diag}(-\lambda_0, \ldots, -\lambda_N, \lambda_0, \ldots, \lambda_N). \tag{5.3}$$

Of course, $\mathscr{S}$ has the block structure (2.7), although the defining equations are more easily written down using the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ introduced in (4.21) rather than $\boldsymbol{G}$ and $\boldsymbol{H}$. We get the matrix equations

$$\boldsymbol{\Phi}\boldsymbol{F} = -\boldsymbol{\Lambda}\boldsymbol{\Psi} \qquad \boldsymbol{\Psi}\boldsymbol{F}' = -\boldsymbol{\Lambda}\boldsymbol{\Phi} \tag{5.4}$$

where $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_0, \ldots, \lambda_N)$ and $\boldsymbol{F}$ is a sparse matrix whose non-zero entries are

$$\boldsymbol{F}_{m,m} = 2km \quad 0 < m < N \qquad \boldsymbol{F}_{m,m-1} = (2m-1) \quad 0 < m \leqslant N. \tag{5.5}$$

Subsequent to solving the Hamiltonian problem, the eigenvalues $\gamma_j(u, k)$ of the CTM are given by the exponential relation

$$\gamma_j(u, k) = u\lambda_j(k). \tag{5.6}$$

This simplification is quite dramatic even for numerical purposes: here we employ it to obtain an analytic solution for the eigenvectors.

Define generating functions of the eigenvectors, $\Phi_j(t)$ and $\Psi_j(t)$, by

$$\Phi_j(t) = \sum_{m=0}^{\infty} \phi_j(m) t^m \qquad \Psi_j(t) = \sum_{m=0}^{\infty} \psi_j(m) t^m. \tag{5.7}$$

The technical details which lead to formulae for $\Phi_j(t)$ and $\Psi_j(t)$ are very similar to those given in [19], and will be omitted. After introducing the new variable $v$ via $t=-k\,sn^2\,v$, we find for $j\geq 1$

$$\Phi_j(v)=k\rho_j\,\mathrm{sd}\,v\,\sin[(j-\tfrac{1}{2})\pi v/K] \qquad \Psi_j(v)=\rho_j\,\mathrm{nc}\,v\,\cos[(j-\tfrac{1}{2})\pi v/K]$$

$$\rho_j^2=\frac{2\pi}{kK}\frac{q^{j-1/2}}{1+q^{2j-1}} \qquad q=\exp(-\pi K'/K) \tag{5.8}$$

while the corresponding eigenvalues attain the limiting values

$$\lambda_j(k)=(j-\tfrac{1}{2})\pi/K. \tag{5.9}$$

The generating functions are even periodic functions of the variable $v$ so it is natural to use a Fourier cosine decomposition to define a new coordinate basis. Effectively we go to a momentum space representation in this way, a point of view which is discussed by Thacker and Itoyama [17] in connection with the Visaroro algebra of non-critical CTMs. Explicitly, we define a new set of coefficients $\hat{\phi}_j(l)$ and $\hat{\psi}_j(l)$ by the expansions

$$\Phi_j(v)=\sum_{l=0}^{\infty}\hat{\phi}_j(l)\cos(l\pi v/K) \qquad \Psi_j(v)=\sum_{l=0}^{\infty}\hat{\psi}_j(l)\cos(l\pi v/K). \tag{5.10}$$

The coefficients may be obtained in explicit form from standard series expansions of Jacobian elliptic functions sd and nc [20], together with trigonometric addition formulae. We get

$$\Phi_j(v)=\frac{\pi\rho_j}{k'K}\sum_{n=0}^{j-1}{}'(-1)^{j-n-1}\cos(n\pi v/K)-\frac{\pi\rho_j}{k'K}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}}{1+q^{2n+1}}$$
$$\times\left[\cos\frac{(n+j)\pi v}{K}+\cos\frac{(n-j+1)\pi v}{K}\right] \tag{5.11}$$
$$\Psi_j(v)=\frac{\pi\rho_j}{k'K}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n+1/2}}{1+q^{2n+1}}\left[\cos\frac{(n+j)\pi v}{K}-\cos\frac{(n-j+1)\pi v}{K}\right]$$

where the prime on the summation means that the $n=0$ term is given weight one-half. From (5.11) the Fourier coefficients in (5.10) may be read off.

Now multiply equations (4.22) for the null vector by $t^m$ and sum over $m\geq 0$. The left-hand side simply changes through the replacement of the coefficients $\phi_j(m)$ and $\psi_j(m)$ by the corresponding generating functions. The right-hand side transforms to $\sum_{m=0}^{\infty}\delta_{m,0}t^m=1$. If we expand the resulting functional equation in cosine functions, the left-hand side involves the coefficients $\hat{\phi}_j(l)$ and $\hat{\psi}_j(l)$ whilst the Fourier coefficient for the right-hand side is $\delta_{l,0}$. Therefore we may write down the equations which determine the components $w_j$ of the null vector, for $1\leq j<\infty$, in a new form appropriate to the thermodynamic limit, viz

$$\sum_{j=1}^{\infty}w_j\{\cosh[(j-\tfrac{1}{2})\pi u/K]\hat{\psi}_j(l)-\sinh[(j-\tfrac{1}{2})\pi u/K]\hat{\phi}_j(l)\}=\delta_{l,0}. \tag{5.12}$$

These equations, which are now explicitly stated in terms of the natural variables $u$ and $k$ of the problem, give the magnetization through the fact that $w$ is normalized: $w_0^2=1-\sum_{j=1}^{\infty}w_j^2$.

## 6. Numerical and asymptotic results

We have not found a general analytical solution of equation (4.22) or (5.12) but they can be solved numerically. It is important to notice the difference between the two. The form (4.22) is for a finite system and the index $m$ is a position coordinate. If the eigenvalues and eigenvectors are found by direct diagonalization of the matrix $\mathscr{A}$, the result is exactly the magnetization for this finite system. On the other hand, the form (5.12) is for an infinite system, and the index $l$ is a momentum coordinate: truncating the infinite system at $L$ is a momentum space cut-off. For either choice, the value of $N$ (respectively $L$) has to be increased until the desired accuracy is attained: this size also depends on the values of the variables $u$ and $k$. Equation (5.12) is particularly convenient for two reasons: first, the necessary matrices may be constructed directly from the explicit formulae of section 5; second, the solution for the null vector converges rapidly with increasing $L$, even near the critical point. For example, it is not necessary to go beyond $L \approx 100$ even when $k' \approx 10^{-5}$. The only numerical problem which is encountered is that $m_c$ is calculated as the difference between 1 and $\Sigma_{j=1}^{\infty} w_j^2$: whilst the components of the vector and the sum both converge rapidly with increasing $L$, there is the usual loss of accuracy when the difference is taken. Even so, computation of $m_c$ to (say) five-figure accuracy is a simple problem over a very wide range of parameter values.

Results are given in figure 2, where the corner magnetization of an isotropic system with opening angles $\theta = 90°, 180°, 279°, 360°$ is plotted against the temperature, measured by $k$. These four cases correspond exactly to the values $u = K'/2,\ K',\ 3K'/2,\ 2K'$ respectively. The curve for $\theta = 90°$ agrees with the result of [4] which is a useful check on our calculation. Also the curve for $\theta = 180°$ coincides with the analytical formula $m_c = (1-k)^{1/2}$ given in [4]. The two other curves, however, are new. Note that $\theta = 360°$ corresponds to a full plane with a cut emanating from the central spin, not to the bulk result for a full plane. Consequently $m_c$ is below the Onsager bulk value, $(k')^{1/4}$. The behaviour near $T_c$ leads to the well-known corner exponent $\beta_c = \pi/2\theta$ [1, 2]. One could easily treat even larger angles describing a system in the form of a staircase. The magnetization curves then assume a more and more rectangular shape, as one would expect. Note that for a fixed angle of 90°, variation of $u$ in the range $0 < u < K'$

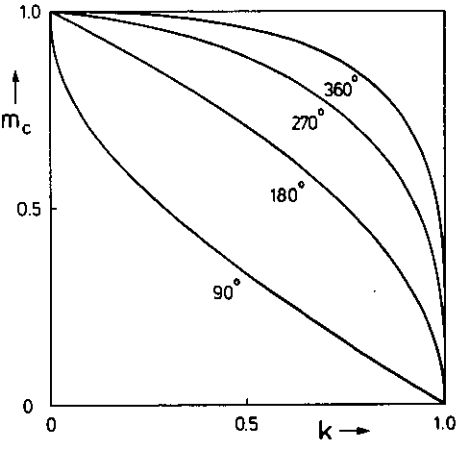


**Figure 2.** Corner magnetization against temperature measured by the Onsager parameter $k$ for isotropic lattices and four different opening angles.

corresponds to variation of the anisotropy. The result for 180°, on the other hand, is obtained by multiplying together $A(u, k)$ and $A(K'-u, k)$ so that $u$ cancels and $m_c$ is independent of the anisotropy.

Although the variable $u$ was introduced to measure the anisotropy, it may also be regarded as an effective angle, through the relation

$$\theta_{\text{eff}} = \pi u / K'. \tag{6.1}$$

At criticality, such an interpretation is quite clear, and has been used in various discussions of conformal properties [1, 12, 13] of corners. We find that the dependence of $m_c$ on $\theta_{\text{eff}}$ has the asymptotic form

$$m_c(\theta_{\text{eff}}) = a(T) \exp[-b(T)/\theta_{\text{eff}}] \qquad \theta_{\text{eff}} \ll 1 \tag{6.2}$$

for all temperatures, not just at the critical temperature. In figure 3, we show the behaviour of $m_c$ as a function of $u$ for small $u$ (because the elliptic quarter period $K'$ has a singularity at $k=0$, this variable is more convenient than $\theta_{\text{eff}}$). As can be seen, this is of the form of equation (6.2) for all values of $k$ down to $k=0$. This is the same feature as found for the apex magnetization at the tip of a cone [1]. There one finds from (3.11) with $\omega_j = \gamma_j(u, k)$, where the latter is given by (5.6) and (5.9), that

$$m_a \approx 2^{1/2} \exp(-\pi K/4u). \tag{6.3}$$

In the present case the constant in the exponent differs from (6.3) by a factor $\alpha$ with $\alpha \to 1$ for $k \to 0$ and $\alpha \to 2$ for $k \to 1$. Near the critical temperature, where $K \sim \ln(T_c - T)$, this formula gives the corner exponent as $\beta_c = \pi/2\theta_{\text{eff}}$ reflecting the anisotropy of the system [1, 13].

We may obtain this asymptotic information, at least for low temperature, from the fact that equations (4.22) are trivally solved when $k=0$. For the eigenvectors and

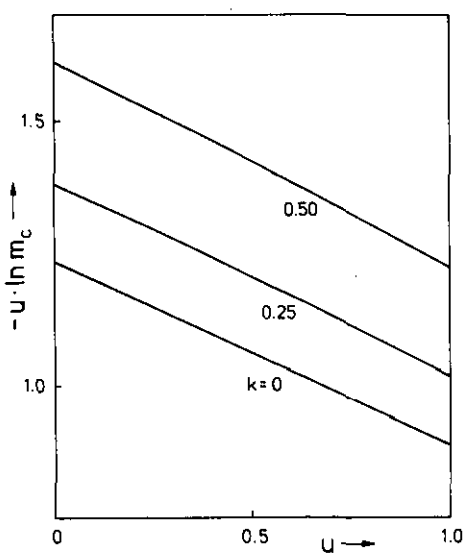


**Figure 3.** Dependence of the corner magnetization on the anisotropy parameter $u$, for three different temperatures. The linearity of the graphs for small $u$ corresponds to the asymptotic forms (6.2) and (6.3).

eivenvalues we have

$$\begin{aligned} &\phi_j(m) = \delta_{j,m} \qquad && \psi_j(m) = \delta_{j,m+1} \pmod N \\ &\lambda_0 = 0 && \lambda_j = 2j-1 \quad j>0 \end{aligned} \tag{6.4}$$

from which it follows that $w_1 = 1/\cosh\frac{1}{2}u$, $w_{j+1} = [\sinh(j-\frac{1}{2})u/\cosh(j+\frac{1}{2})u]w_j$, $1 \leq j < N$, $w_0 = [\sinh(N-\frac{1}{2})u]w_N$, giving

$$m_c = \prod_{j=1}^{N} \tanh(j-\tfrac{1}{2})u. \tag{6.5}$$

For $N \to \infty$, the infinite product is of a familiar type and a complementary nome transformation [9] gives the asymptotic form (6.3) (with $K = \pi/2$) for small $u$.

## 7. Summary and outlook

In this paper we have treated the corner magnetization problem in a new way using special properties of Baxter's corner transfer matrices. After a number of steps, some of which involved unexpected simplifications, we arrived at a system of linear equations, in either a coordinate or momentum representation, from which $m_c$ is determined in a very simple way. In this sense the situation is similar to that encountered in the treatment of a 90° corner by the row-to-row transfer matrix [1, 4, 5]. However, the present technique is much more adapted to the corner problem since one can handle various opening angles in a unified formulation.

The equations can be solved numerically for all temperatures. In particular, the form (5.12) turns out to be a very efficient way to obtain the thermodynamic limiting values of $m_c$ using rather small matrices, even very close to the critical temperature. It also gives an indication of why the case $\theta = 180°$ is special: it brings about certain simplifications in the coefficients of the matrix equation. This is due to the fact that in this case the natural variable $x = \exp(-\pi u/2K)$, which incorporates the effective angle, is equal to the elliptic nome $q$. Still, even for $\theta = 180°$ we did not succeed in obtaining the null vector analytically, and thereby re-deriving the exact formula for $m_c$. Moreover, the special nature of the equations when $\theta = 180°$ makes it likely that there is no simple formula for $m_c$ for other values of $\theta_{\text{eff}}$. This must await further investigation.

One should also be able to extend the techniques of this paper to the other type of corner in a square lattice, where the free edges are along the directions of the bonds. Although in this case the exponent for $\theta = 90°$ is independent of the anisotropy, there is an interesting conjecture for the dependence of $m_c$ on the interaction strengths [5]. Similarly, the 60° corners in a triangular lattice could be treated. Finally, the more complicated solvable models are natural candidates for analogous investigations. Since in these cases even the bulk order parameters can be found only with the help of CTMs, one should expect CTMs to be needed for an investigation of their order parameters at corners, including straight surfaces.

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